Numbers Exam

Block 1A, 2025–2026
October 30, 2025, 11:45-13:45

university of groningen

INSTRUCTIONS TO CANDIDATES

1. This exam consists of six problems and a bonus problem. The total number of points available is 100.
2. The number of points you can get for each question is shown below.
3. In answering the questions in this paper it is particularly important to show your argumentation. The total number of points will only be given for full and detailed answers.
4. You are not allowed to use a "cheat sheet" or any notes, books, or other materials during the exam.
5. You may use a basic calculator (optional), but programmable or graphing calculators and any other computing devices are strictly prohibited.
6. Please write your solutions in the space provided below each problem on the accompanying pages, and ensure that your name and student ID are on every page.

DO NOT REMOVE THIS DOCUMENT FROM THE EXAMINATION ROOM.

| Att | Q1 | Q2 | Q3 | Q4 | Q5 | Q6 | Bonus | Total |
|---|---|---|---|---|---|---|---|---|
| | | | | | | | | |
| **10 pt** | **10 pt** | **8 pt** | **24 pt** | **14 pt** | **14 pt** | **20 pt** | **5 pt** | **100 pt** |

**Full Name:** .....

**Student Number:** .....

**1** Prove that $2^{n+2}$ divides $(2n+3)!$ for all positive integers $n$.

**2** Let $a, b$, and $c$ be integers. If $\gcd(a,b) = 1$ and $c \mid (a+b)$, then show that $\gcd(a,c) = 1$.

**3** **(a)** For which values of $k$ does the equation $123x+360y = k$ have at least one integer solution?

**(b)** Solve the linear congruence $360x \equiv 99 \pmod{123}$, or explain why no solutions exist.

**4** A *Carmichael number* is an odd composite number $n$ which satisfies $a^{n-1} \equiv 1 \pmod{n}$ for every integer $a$ with $\gcd(a, n) = 1$. Show that $561 = 3 \cdot 11 \cdot 17$ is a Carmichael number.

5 **(a)** State Fermat's (little) theorem.

**(b)** Let $p$ and $q$ be distinct primes and let $a$ be an integer such that $\gcd(a, pq) = 1$. If $a^p \equiv a \pmod{q}$ and $a^q \equiv a \pmod{p}$, then show that $a^{pq} \equiv a \pmod{pq}$.

**6** For each of the following four statements, prove it if it is correct, and disprove it if it is incorrect.

**(a)** If $m$ is a positive integer and $a^2 \equiv b^2 \pmod{m}$, then $a \equiv b \pmod{m}$ or $a \equiv -b \pmod{m}$.

**(b)** $\{\, 5x + 12y + 8z \mid x, y, z \in \mathbb{Z} \,\} = \mathbb{Z}$.

**(c)** $2^{11} + 1$ divides $2^{66} + 1$.

**(d)** Over $\mathbb{Z}/39\mathbb{Z}$ the matrix equation $\begin{bmatrix} \overline{5} & \overline{3} \\ \overline{3} & \overline{12} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \overline{4} \\ \overline{5} \end{bmatrix}$ has a unique solution.

**Bonus.** For distinct primes $p$ and $q$, show that

$$\varphi(p^e q^f) = \varphi(p^e)\,\varphi(q^f)$$

for any positive integers $e$ and $f$, where $\varphi$ denotes Euler's totient function.